Chapter 6

Techniques of Integration

6.1 Integration by formulae

There exist many books that contain extensive lists of integration, differentiation and other mathematical formulae. For our purpose we will use the list given below.

1. $\int af(u)du = a\int f(u)du$

2. $\int \left(\sum_{i=1}^{n} a_i f_i(u)\right) du = \sum_{i=1}^{n} \left(\int a_i f_i(u)du\right)$

3. $\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

4. $\int u^{-1} du = \ln|u| + C$

5. $\int e^{au} du = \frac{e6au}{a} + C$

6. $\int a^{bu} du = \frac{a^{bu}}{b \ln a} + C, \; a > 0, \; a \neq 1$

7. $\int \ln|u| du = u \ln|u| - u + C$

8. $\displaystyle\int \sin(au)du = \frac{-\cos(au)}{a} + C$

9. $\displaystyle\int \cos(au)du = \frac{\sin(au)}{a} + C$

10. $\displaystyle\int \tan(au)du = \frac{\ln|\sec(au)|}{a} + C$

11. $\displaystyle\int \cot(au)du = \frac{\ln|\sin(au)|}{a} + C$

12. $\displaystyle\int \sec(au)du = \frac{\ln|\sec(au) + \tan(au)|}{a} + C$

13. $\displaystyle\int \csc(au)du = \frac{\ln|\csc(au) - \cot(au)|}{a} + C$

14. $\displaystyle\int \sinh(au)du = \frac{\cosh(au)}{a} + C$

15. $\displaystyle\int \cosh(au)du = \frac{\sinh(au)}{a} + C$

16. $\displaystyle\int \tanh(au)du = \frac{\ln|\cosh(au)|}{a} + C$

17. $\displaystyle\int \coth(au)du = \frac{\ln|\sinh(au)|}{a} + C$

18. $\displaystyle\int \operatorname{sech}(au)du = \frac{2}{a}\ \arctan(e^{au}) + C$

19. $\displaystyle\int \operatorname{csch}(au)\ du = \frac{2}{a}\ \operatorname{arctanh}(e^{au}) + C$

20. $\displaystyle\int \sin^2(au)du = \frac{u}{2} - \frac{\sin(au)\cos(au)}{2a} + C$

21. $\displaystyle\int \cos^2(au)du = \frac{u}{2} + \frac{\sin(au)\cos(au)}{2a} + C$

22. $\displaystyle\int \tan^2(au)du = \frac{\tan(au)}{a} - u + C$

23. $\displaystyle\int \cot^2(au)du = -\frac{\cot(au)}{a} - u + C$

24. $\displaystyle\int \sec^2(au)du = \frac{\tan(au)}{a} + C$

25. $\displaystyle\int \csc^2(au)du = -\frac{\cot(au)}{a} + C$

26. $\displaystyle\int \sinh^2(au)du = -\frac{u}{2} + \frac{\sinh(2au)}{4a} + C$

27. $\displaystyle\int \cosh^2(au)du = \frac{u}{2} + \frac{\sinh(2au)}{4a} + C$

28. $\displaystyle\int \tanh^2(au)du = u - \frac{\tanh(au)}{a} + C$

29. $\displaystyle\int \coth^2(au)du = u - \frac{\coth(au)}{a} + C$

30. $\displaystyle\int \operatorname{sech}^2(au)du = \frac{\tanh(au)}{a} + C$

31. $\displaystyle\int \operatorname{csch}^2(au)du = \frac{-\coth(au)}{a} + C$

32. $\displaystyle\int \sec(au)\tan(au)du = \frac{\sec(au)}{a} + C$

33. $\displaystyle\int \csc(au)\cot(au)du = -\frac{\csc(au)}{a} + C$

34. $\displaystyle\int \operatorname{sech}(au)\tanh(au)du = -\frac{\operatorname{sech}(au)}{a} + C$

35. $\displaystyle\int \operatorname{csch}(au)\coth(au)du = -\frac{\operatorname{csch}(au)}{a} + C$

36. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\ \arctan\left(\frac{u}{a}\right) + C$

37. $\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{a}\ \operatorname{arctanh}\left(\frac{u}{a}\right) + C = \frac{1}{2a}\ \ln\left|\frac{a+u}{a-u}\right| + C$

38. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \operatorname{arcsinh}\left(\frac{u}{a}\right) + C$

39. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C,\ |a| > |u|$

40. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \operatorname{arccosh}\left(\frac{u}{a}\right) + C,\ |u| > |a|$

41. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\ \operatorname{arcsec}\left(\frac{u}{a}\right) + C,\ |u| > |a|$

42. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ \operatorname{arcsech}\left(\frac{u}{a}\right) + C,\ |a| > |u|$

43. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ \operatorname{arccsch}\left(\frac{u}{a}\right) + C$

44. $\displaystyle\int \frac{u\ du}{\sqrt{a^2+u^2}} = \sqrt{a^2+u^2} + C$

45. $\displaystyle\int \frac{u\ du}{a^2-u^2} = -\ln\sqrt{a^2-u^2} + C,\ |a| > |u|$

46. $\displaystyle\int \frac{u\ du}{\sqrt{a^2+u^2}} = \sqrt{a^2+u^2} + C$

47. $\displaystyle\int \frac{u\ du}{\sqrt{a^2-u^2}} = -\sqrt{a^2-u^2} + C,\ |a| > |u|$

48. $\displaystyle\int \frac{u\ du}{\sqrt{u^2-a^2}} = \sqrt{u^2-a^2} + C,\ |u| > |a|$

49. $\displaystyle\int \arcsin(au)du = u\arcsin(au) + \frac{1}{a}\ \sqrt{1-a^2u^2} + C,\ |a||u| < 1$

50. $\displaystyle\int \arccos(au)du = u\arccos(au) - \frac{1}{a}\ \sqrt{1-a^2u^2} + C,\ |a||u| < 1$

51. $\displaystyle\int \arctan(au)du = u\arctan(au) - \frac{1}{2a}\ \ln(1+a^2u^2) + C$

52. $\displaystyle\int \operatorname{arccot}(au)du = u\operatorname{arccot}(au) + \frac{1}{2a}\ \ln(1+a^2u^2) + C$

53. $\displaystyle\int \operatorname{arcsec}(au)du = u\operatorname{arcsec}(au) - \frac{1}{a}\ \ln\left|au + \sqrt{a^2u^2-1}\right| + C,\ au > 1$

54. $\displaystyle\int \operatorname{arccsc}(au)du = u\operatorname{arccsc}(au) + \frac{1}{a}\ \ln\left|au + \sqrt{a^2u^2-1}\right| + C,\ au > 1$

55. $\displaystyle\int \operatorname{arcsinh}(au)du = u\operatorname{arcsinh}(au) - \frac{1}{a}\ \sqrt{1+a^2u^2} + C$

56. $\displaystyle\int \operatorname{arccosh}(au)du = u\operatorname{arccosh}(au) - \frac{1}{a}\ \sqrt{-1+a^2u^2} + C,\ |a||u| > 1$

57. $\displaystyle\int \operatorname{arctanh}(au)du = u\operatorname{arctanh}(au) + \frac{1}{2a}\ \ln(-1+a^2u^2) + C,\ |a||u| \neq 1$

58. $\displaystyle\int \operatorname{arccoth}(au)du = u\operatorname{arccoth}(au) + \frac{1}{2a}\ \ln(-1+a^2u^2) + C,\ |a||u| \neq 1$

59. $\displaystyle\int \operatorname{arcsech}(au)du = u\operatorname{arcsech}(au) + \frac{1}{a}\ \arcsin(au) + C,\ |a||u| < 1$

60. $\displaystyle\int \operatorname{arccsch}(au)du = u\operatorname{arccsch}(au) + \frac{1}{a}\ \ln\left|au + \sqrt{a^2u^2+1}\right| + C$

61. $\displaystyle\int e^{au}\sin(bu)du = \frac{e^{au}[a\sin(bu) - b\cos(bu)]}{a^2+b^2} + C$

62. $\displaystyle\int e^{au}\cos(bu)du = \frac{e^{au}[a\cos(bu) + b\sin(bu)]}{a^2+b^2} + C$

63. $\displaystyle\int \sin^n(u)du = \frac{-1}{n}\ \left[\sin^{n-1}(u)\cos(u)\right] + \frac{n-1}{n}\ \int \sin^{n-2}(u)du$

64. $\displaystyle\int \cos^n(u)du = \frac{1}{n}\ \left[\cos^{n-1}(u)\sin(u)\right] + \frac{n-1}{n}\int \cos^{n-2}(u)du$

65. $\displaystyle\int \tan^n(u)du = \frac{\tan^{n-1}(u)}{n-1} - \int \tan^{n-2}(u)du$

66. $\displaystyle\int \cot^n(u)du = -\frac{\cot^{n-1}(u)}{n-1} - \int \cot^{n-2}(u)du$

67. $\displaystyle\int \sec^n(u)du = \frac{1}{n-1}\ \left[\sec^{n-2}(u)\tan(u)\right] + \frac{n-2}{n-1}\int \sec^{n-2}(u)du$

68. $\displaystyle\int \csc^n(u)du = \frac{-1}{n-1}\left[\csc^{n-2}(u)\cot(u)\right] + \frac{n-2}{n-1}\int \csc^{n-2}(u)du$

69. $\displaystyle\int \sin(mu) sin(nu)du = \frac{\sin[(m-n)u]}{2(m-n)} - \frac{\sin[(m+n)u]}{2(m+n)} + C,\ m^2 \neq n^2$

70. $\displaystyle\int \cos(mu)\cos(nu)du = \frac{\sin[(m-n)u]}{2(m-n)} + \frac{\sin[(m+n)u]}{2(m+n)} + C,\ m^2 \neq n^2$

71. $\displaystyle\int \sin(mu)\cos(nu)du = \frac{\cos[(m-n)u]}{2(m-n)} - \frac{\cos[(m+n)u]}{2(m+n)} + C,\ m^2 \neq n^2$

Exercises 6.1

1. Define the statement that $g(x)$ is an antiderivative of $f(x)$ on the closed interval $[a, b]$

2. Prove that if $g(x)$ and $h(x)$ are any two antiderivatives of $f(x)$ on $[a, b]$, then there exists some constant C such that $g(x) = \ln(x) + C$ for all x on $[a, b]$.

In problems 3–30, evaluate each of the indefinite integrals.

3. $\displaystyle\int x^5 dx$

4. $\displaystyle\int \frac{4}{x^3}\, dx$

5. $\displaystyle\int x^{-3/5} dx$

6. $\displaystyle\int 3x^{2/3} dx$

7. $\displaystyle\int \frac{2}{\sqrt{x}}\, dx$

8. $\displaystyle\int t^2\sqrt{t}\; dt$

9. $\displaystyle\int \left(t^{-1/2} + t^{3/2}\right) dt$

10. $\displaystyle\int (1+x^2)^2 dx$

11. $\displaystyle\int t^2(1+t)^2 dt$

12. $\displaystyle\int (1+t^2)(1-t^2)dt$

13. $\displaystyle\int \left(\frac{1}{t^{1/2}} + \sin t\right) dt$

14. $\displaystyle\int (2\sin t + 3\cos t)dt$

15. $\displaystyle\int 3\sec^2 t\; dt$

16. $\displaystyle\int 2\csc^2 x\; dx$

17. $\displaystyle\int 4\sec t\tan t\; dt$

18. $\displaystyle\int 2\csc t\cot t\; dt$

19. $\displaystyle\int \sec t(\sec t + \tan t)dt$

20. $\int \csc t(\csc t - \cot t)dt$

21. $\int \frac{\sin x}{\cos^2 x}\, dx$

22. $\int \frac{\cos x}{\sin^2 x}\, dx$

23. $\int \frac{\sin^3 t - 3}{\sin^2 t}\, dt$

24. $\int \frac{\cos^3 t + 2}{\cos^2 t}\, dt$

25. $\int \tan^2 t\ dt$

26. $\int \cot^2 t\ dt$

27. $\int (2\sec^2 t + 1)dt$

28. $\int \frac{2}{t}\, dt$

29. $\int \sinh t\ dt$

30. $\int \cosh t\ dt$

31. Determine $f(x)$ if $f'(x) = \cos x$ and $f(0) = 2$.

32. Determine $f(x)$ if $f''(x) = \sin x$ and $f(0) = 1,\ f'(0) = 2$.

33. Determine $f(x)$ if $f''(x) = \sinh x$ and $f(0) = 2,\ f'(0) = -3$.

34. Prove each of the integration formulas 1–77.

6.2 Integration by Substitution

Theorem 6.2.1 *Let $f(x), g(x), f(g(x))$ and $g'(x)$ be continuous on an interval $[a, b]$. Suppose that $F'(u) = f(u)$ where $u = g(x)$. Then*

(i) $\int f(g(x))g'(x)dx = \int f(u)du = F(g(x)) + C$

(ii) $\int_a^b f(g(x))g'(x)dx = \int_{u=g(u)}^{u=g(b)} f(u)du = F(g(b)) - F(g(a)).$

Proof. See the proof of Theorem 5.3.1.

Exercises 6.2 In problems 1–39, evaluate the integral by making the given substitution.

1. $\int 3x(x^2+1)^{10}dx,\ u = x^2+1$

2. $\int x\sin(1+x^2)dx,\ u = 1+x^2$

3. $\int \frac{\cos(\sqrt{t})}{\sqrt{t}}\,dt,\ x = \sqrt{t}$

4. $\int \frac{3x^2}{(1+x^3)^{3/2}}\,dx,\ u = 1+x^3$

5. $\int \frac{2e^{\arcsin x}}{\sqrt{1-x^2}}\,dx,\ u = \arcsin x$

6. $\int \frac{3e^{\arccos x}}{\sqrt{1-x^2}}\,dx$

7. $\int x\ 4^{x^2}dx,\ u = 4^{x^2}$

8. $\int 10^{\sin x}\cos x\ dx,\ u = \sin x$

9. $\int \frac{4^{\arctan x}}{1+x^2}\,dx,\ u = 4^{\arctan x}$

10. $\int \frac{(1+\ln x)^{10}}{x}\,dx,\ u = 1+\ln x$

11. $\int \frac{5^{\operatorname{arcsec} x}}{x\sqrt{x^2-1}}\,dx,\ u = \operatorname{arcsec} x$

12. $\int (\tan 2x)^3\sec^2 2x\ dx,\ u = \tan 2x$

13. $\int (\cot 3x)^5\csc^2 3x\ dx,\ u = \cot 3x$

14. $\int \sin^{21} x\cos x\ dx,\ u = \sin x$

15. $\int \cos^5 x\sin x\ dx,\ u = \cos x$

16. $\int (1+\sin x)^{10}\cos x\ dx,\ u = 1+\sin x$

17. $\int \sin^3 x\ dx,\ u = \cos x$

18. $\int \cos^3 x\ dx,\ u = \sin x$

19. $\int \tan^3 x\ dx,\ u = \tan x$

20. $\int \cot^3 x\ dx,\ u = \cot x$

21. $\int \sec^4 x\ dx,\ u = \tan x$

22. $\int \csc^4 x\ dx,\ u = \cot x$

23. $\int \sin^3 x\cos^3 x\ dx,\ u = \sin x$

24. $\int \sin^3 x\cos^3 x\ dx,\ u = \cos x$

25. $\int \tan^4 x\ dx,\ u = \tan x$

26. $\int \frac{\sin(\ln x)}{x}\,dx,\ u = \ln x$

27. $\displaystyle\int \frac{x\cos(\ln(1+x^2))}{1+x^2}\,dx,\ u=\ln(1+x)^2$

28. $\displaystyle\int \tan^3 x\sec^4 x\ dx,\ u=\sec x$

29. $\displaystyle\int \cot^3 x\csc^4 x\ dx,\ u=\csc x$

30. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}, x=2\sin t$

31. $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}, x=3\cos t$

32. $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}, x=2\sinh t$

33. $\displaystyle\int \frac{dx}{\sqrt{x^2-9}}, x=3\cosh t$

34. $\displaystyle\int \frac{dx}{4+x^2}, x=2\tan t$

35. $\displaystyle\int \frac{dx}{4-x^2}, x=2\tanh t$

36. $\displaystyle\int \frac{dx}{x\sqrt{x^2-4}}, x=2\sec t$

37. $\displaystyle\int 4e^{\sin(3x)}\cos(3x)dx,\ u=\sin 3x$

38. $\displaystyle\int x\ 3^{(x^2+4)}dx,\ u=3^{x^2+4}$

39. $\displaystyle\int 3\ e^{\tan 2x}\sec^2 x\ dx,\ u=\tan 2x$

40. $\displaystyle\int x\sqrt{x+2}\ dx,\ u=x+2$

Evaluate the following definite integrals.

41. $\displaystyle\int_0^1 (x+1)^{30}dx$

42. $\displaystyle\int_1^2 x(4-x^2)^{1/2}dx$

43. $\displaystyle\int_0^{\pi/4} \tan^3 x\sec^2 x\ dx$

44. $\displaystyle\int_0^1 x^3(x^2+1)^3dx$

45. $\displaystyle\int_0^2 (x+1)(x-2)^{10}dx$

46. $\displaystyle\int_0^8 x^2(1+x)^{1/2}dx$

47. $\displaystyle\int_0^{\pi/6} \sin(3x)dx$

48. $\displaystyle\int_0^{\pi/4} \cos(2x)dx$

49. $\displaystyle\int_0^{\pi/4} \sin^3 2x\cos 2x\ dx$

50. $\displaystyle\int_0^{\pi/6} \cos^4 3x\sin 3x\ dx$

51. $\displaystyle\int_0^1 \frac{e^{\arctan x}}{1+x^2}\ dx$

52. $\displaystyle\int_0^{1/2} \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\ dx$

53. $\displaystyle\int_2^3 \frac{e^{\operatorname{arcsec} x}}{x\sqrt{x^2-1}}\, dx$

54. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^2}}$

6.3 Integration by Parts

Theorem 6.3.1 *Let $f(x), g(x), f'(x)$ and $g'(x)$ be continuous on an interval $[a, b]$. Then*

(i) $\displaystyle\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$

(ii) $\displaystyle\int_a^b f(x)g'(x)dx = (f(b)g(b) - f(a)g(a)) - \int_a^b g(x)f'(x)dx$

(iii) $\displaystyle\int udv = uv - \int vdu$

where $u = f(x)$ and $dv = g'(x)dx$ are the parts of the integrand.

Proof. See the proof of Theorem 5.4.1.

Exercises 6.3 Evaluate each of the following integrals.

1. $\displaystyle\int x \sin x\ dx$

2. $\displaystyle\int x \cos x\ dx$

3. $\displaystyle\int x \ln x\ dx$

4. $\displaystyle\int x\ e^x\ dx$

5. $\displaystyle\int x\ 4^x\ dx$

6. $\displaystyle\int x^2 \ln x\ dx$

7. $\displaystyle\int x^2 \sin x\ dx$

8. $\displaystyle\int x^2 \cos x\ dx$

9. $\displaystyle\int x^2 e^x\ dx$

10. $\displaystyle\int x^2\ 10^x\ dx$

11. $\int e^x \sin x \; dx$ (Let $u = e^x$ twice and solve.)

12. $\int e^x \cos x \; dx$ (Let $u = e^x$ twice and solve.)

13. $\int e^{2x} \sin 3x \; dx$ (Let $u = e^{2x}$ twice and solve.)

14. $\int x \sin(3x) dx$

15. $\int x^2 \cos(2x) dx$

16. $\int x^2 e^{4x} dx$

17. $\int x^3 \ln(2x) dx$

18. $\int x \sec^2 x \; dx$

19. $\int x \csc^2 x \; dx$

20. $\int x \sinh(4x) dx$

21. $\int x^2 \cosh x \; dx$

22. $\int x \cos(5x) dx$

23. $\int \sin(\ln x) dx$

24. $\int \cos(\ln x) dx$

25. $\int x \arcsin x \; dx$

26. $\int x \arccos x \; dx$

27. $\int x \arctan x \; dx$

28. $\int x \; \text{arcsec}\, x \; dx$

29. $\int \arcsin x \; dx$

30. $\int \arccos x \; dx$

31. $\int \arctan x \; dx$

32. $\int \text{arcsec}\, x \; dx$

Verify the following integration formulas:

33. $\displaystyle\int \sin^n(ax)dx = -\frac{\sin^{n-1}(ax)\cos(ax)}{na} + \frac{n-1}{n}\int(\sin^{n-2} ax)dx$

34. $\displaystyle\int \cos^n(ax)dx = \frac{1}{na}\ \cos^{n-1}(ax)\sin(ax) + \frac{n-1}{n}\int(\cos^{n-2} ax)dx$

35. $\displaystyle\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx$

36. $\displaystyle\int x^n \sin x\ dx = -x^n\cos x + n\int x^{n-1}\cos x\ dx$

37. $\displaystyle\int x^n \cos x\ dx = x^n\sin x - n\int x^{n-1}\sin x\ dx$

38. $\displaystyle\int e^{ax}\sin(bx)dx = \frac{1}{a^2+b^2}\ e^{ax}[a\sin(bx) - b\cos(bx)] + C$

39. $\displaystyle\int e^{ax}\cos(bx)\ dx = \frac{1}{a^2+b^2}\ e^{ax}[a\cos(bx) + b\sin(bx)] + C$

40. $\displaystyle\int x^n \ln x\ dx = \frac{1}{n+1}\ x^{n+1}\ln x - \frac{1}{(n+1)^2}\ x^{n+1} + C,\ n \neq -1,\ x > 0$

41. $\displaystyle\int \sec^n x\ dx = \frac{1}{n-1}\ \sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx,\ n \neq 1,\ n > 0$

42. $\displaystyle\int \csc^n x\ dx = \frac{-1}{n-1}\ \csc^{n-2} x\cot x + \frac{n-2}{n-1}\int \csc^{n-2} x\ dx,\ n \neq 1, n > 0$

Use the formulas 33–42 to evaluate the following integrals:

43. $\displaystyle\int \sin^4 x\ dx$

44. $\displaystyle\int \cos^5 x\ dx$

45. $\displaystyle\int x^3 e^x dx$

46. $\displaystyle\int x^4 \sin x\ dx$

47. $\displaystyle\int x^3 \cos x\ dx$

48. $\displaystyle\int e^{2x}\sin 3x\ dx$

49. $\displaystyle\int e^{3x}\cos 2x\ dx$

50. $\displaystyle\int x^5 \ln x\ dx$

51. $\int \sec^3 x \; dx$

52. $\int \csc^3 x \; dx$

Prove each of the following formulas:

53. $\int \tan^n x \; dx = \frac{1}{n-1} \; \tan^{n-1} x - \int \tan^{n-2} x \; dx, \; n \neq 1$

54. $\int \cot^n x \; dx = \frac{1}{n-1} \; \cot^{n-1} x - \int \cot^{n-2} x \; dx, \; n \neq 1$

55. $\int \sin^{2n+1} x \; dx = -\int (1-u^2)^n du, \; u = \cos x$

56. $\int \cos^{2n+1} x \; dx = -\int (1-u^2)^n du, \; u = \sin x$

57. $\int \sin^{2n+1} x \; \cos^m x \; dx = -\int (1-u^2)^n u^m du, \; u = \cos x$

58. $\int \cos^{2n+1} x \; \sin^m x \; dx = \int (1-u^2)^n u^m du, \; u = \sin x$

59. $\int \sin^{2n} x \; \cos^{2m} x \; dx = \int (\sin x)^{2n} (1-\sin^2 x)^m dx$

60. $\int \tan^n x \; \sec^{2m} x \; dx = \int u^n (1+u^2)^{m-1} du, \; u = \tan x$

61. $\int \cot^n x \; \csc^{2m} x \; dx = -\int u^n (1+u^2)^{m-1} du, \; u = \cot x$

62. $\int \tan^{2n+1} x \; \sec^m x \; dx = \int (u^2-1)^n u^{m-1} du, \; u = \sec x$

63. $\int \cot^{2n+1} x \; \csc^m x \; dx = -\int (u^2-1)^n u^{m-1} du, \; u = \csc x$

64. $\int \sin mx \cos nx \; dx = -\frac{1}{2} \left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n} \right] + C; \; m^2 \neq n^2$

65. $\displaystyle\int \sin mx \sin nx \; dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right] + C; \; m^2 \neq n^2$

66. $\displaystyle\int \cos mx \cos nx \; dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right] + C; \; m^2 \neq n^2$

6.4 Trigonometric Integrals

The trigonometric integrals are of two types. The integrand of the first type consists of a product of powers of trigonometric functions of x. The integrand of the second type consists of $\sin(nx)\cos(mx), \sin(nx)\sin(mx)$ or $\cos(nx)\cos(mx)$. By expressing all trigonometric functions in terms of sine and cosine, many trigonometric integrals can be computed by using the following theorem.

Theorem 6.4.1 *Suppose that m and n are integers, positive, negative, or zero. Then the following reduction formulas are valid:*

1. $\displaystyle\int \sin^n x \; dx = \frac{-1}{n} \; \sin^{n-1} x \cos x + \frac{(n-1)}{n} \int \sin^{n-2} x \; dx, \; n > 0$

2. $\displaystyle\int \sin^{n-2} x \; dx = \frac{1}{n-1} \; \sin^{n-1} x \cos x + \frac{n}{n-1} \int \sin^n x \; dx, \; n \leq 0$

3. $\displaystyle\int (\sin x)^{-1} \; dx = \int \csc x \; dx = \ln|\csc x - \cot x| + c \text{ or } - \ln|\csc x + \cot x| + c$

4. $\displaystyle\int \cos^n x \; dx = \frac{1}{n} \; \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \; dx, \; n > 0$

5. $\displaystyle\int \cos^{n-2} x \; dx = \frac{-1}{n-1} \; \cos^{n-1} x \sin x + \frac{n}{n-1} \int \cos^n x \; dx, \; n \leq 0$

6. $\displaystyle\int (\cos x)^{-1} \; dx = \int \sec x \; dx = \ln|\sec x + \tan x| + c$

7. $$\begin{aligned}\int sin^n x \cos^{2m+1} x \; dx &= \textstyle\int \sin^n x (1-\sin^2 x)^m \cos x \; dx \\ &= \textstyle\int u^n (1-u^2)^m du, \; u = \sin x, \; du = \cos x \; dx\end{aligned}$$

8. $\displaystyle\int sin^{2n+1} x \cos^m x\ dx = \int \cos^m x(1-\cos^2 x)^n \sin x\ dx$
$\displaystyle = -\int u^m(1-u^2)^n du,\ u = \cos x,\ du = -\sin x\ dx$

9. $\displaystyle\int \sin^{2n} x \cos^{2m} x\ dx = \int (1-\cos^2 x)^n \cos^{2m} x\ dx$
$\displaystyle = \int (1-\sin^2 x)^m \sin^{2n} x\ dx$

10. $\displaystyle\int \sin(nx)\cos(mx) dx = \frac{-1}{2}\left[\frac{\cos(m+n)x}{m-n} + \frac{\cos(m-n)x}{m-n}\right] + c,\ m^2 \neq n^2$

11. $\displaystyle\int \sin(mx)\sin(mx)\ dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right] + c,\ m^2 \neq n^2$

12. $\displaystyle\int \cos(mx)\cos(mx)\ dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right] + c,\ m^2 \neq n^2$

Corollary. The following integration formulas are valid:

13. $\displaystyle\int \tan^n u\ du = \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u\ d$

14. $\displaystyle\int \sec^n u\ du = \frac{1}{n-1}\ \sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx$

15. $\displaystyle\int \csc^n u\ du = \frac{-1}{n-1}\ \csc^{n-2} x \cot x + \frac{n-2}{n-1}\ \int \csc^{n-2} x\ dx$

Exercises 6.4 Evaluate each of the following integrals.

1. $\displaystyle\int \sin^5 x\ dx$

2. $\displaystyle\int \cos^4 x\ dx$

3. $\displaystyle\int \tan^5 x\ dx$

4. $\displaystyle\int \cot^4 x\ dx$

5. $\displaystyle\int \sec^5 x\ dx$

6. $\displaystyle\int \csc^4 x\ dx$

7. $\int \sin^5 x \cos^4 x \ dx$

8. $\int \sin^3 x \cos^5 x \ dx$

9. $\int \sin^4 x \cos^3 x \ dx$

10. $\int \sin^2 x \cos^4 x \ dx$

11. $\int \tan^5 x \sec^4 x \ dx$

12. $\int \cot^5 x \csc^4 x \ dx$

13. $\int \tan^4 x \sec^5 x \ dx$

14. $\int \cot^4 x \csc^5 x \ dx$

15. $\int \tan^4 x \sec^4 x \ dx$

16. $\int \cot^4 x \csc^4 x \ dx$

17. $\int \tan^3 x \sec^3 x \ dx$

18. $\int \cot^3 x \csc^3 x \ dx$

19. $\int \sin 2x \cos 3x \ dx$

20. $\int \sin 4x \cos 4x \ dx$

21. $\int \sin 3x \cos 3x \ dx$

22. $\int \sin 2x \sin 3x \ dx$

23. $\int \sin 4x \sin 6x \ dx$

24. $\int \sin 3x \sin 5x \ dx$

25. $\int \cos 3x \cos 5x \ dx$

26. $\int \cos 2x \cos 4x \ dx$

27. $\int \cos 3x \cos 4x \ dx$

28. $\int \sin 4x \cos 4x \ dx$

6.5 Trigonometric Substitutions

Theorem 6.5.1 ($a^2 - u^2$ Forms). *Suppose that* $u = a \sin t, \ a > 0$. *Then*

$$du = a\cos t dt,\ a^2 - u^2 = a^2\cos^2 t,\ \sqrt{a^2-u^2} = a\cos t,\ t = \arcsin(u/a),$$

$$\sin t = \frac{u}{a},\ \cos t = \frac{\sqrt{a^2-u^2}}{a},\ \tan t = \frac{u}{\sqrt{a^2-u^2}},$$

$$\cot t = \frac{\sqrt{a^2-u^2}}{u},\ \sec t = \frac{a}{\sqrt{a^2-u^2}},\ \csc t\ \frac{a}{u}.$$

graph

The following integration formulas are valid:

1. $\displaystyle\int \frac{udu}{a^2-u^2} = -\frac{1}{2}\ \ln|a^2-u^2| + c$

2. $\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ \ln\left|\frac{a-u}{a+u}\right| + c = \frac{1}{a}\ \operatorname{arctanh}\left(\frac{u}{a}\right) + c$

3. $\displaystyle\int \frac{udu}{\sqrt{a^2-u^2}} = -\sqrt{a^2-u^2} + c$

4. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + c$

5. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = \frac{1}{a}\ \ln\left|\frac{a}{u} - \frac{\sqrt{a^2-u^2}}{u}\right| + c$

6. $\displaystyle\int \sqrt{a^2-u^2}\ du = \frac{a^2}{2}\ \arcsin\left(\frac{u}{a}\right) + \frac{1}{2}\ u\sqrt{a^2-u^2} + c$

Proof. The proof of this theorem is left as an exercise.

Theorem 6.5.2 (a^2+u^2 Forms). *Suppose that* $u = a\tan t,\ a > 0$. *Then*

$$du = a\sec^2 t dt, a^2+u^2 = a^2\sec^2 t, \sqrt{a^2+u^2} = a\sec t, t = \arctan\left(\frac{u}{a}\right),$$

$$\sin t = \frac{u}{\sqrt{a^2+u^2}},\ \cos t = \frac{a}{\sqrt{a^2+u^2}},\ \tan t = \frac{u}{a}$$

$$\csc t = \frac{\sqrt{a^2+u^2}}{u},\ \sec t = \frac{\sqrt{a^2+u^2}}{a},\ \cot t = \frac{a}{u}.$$

graph

Proof. The proof of this theorem is left as an exercise.

The following integration formulas are valid:

1. $\displaystyle\int \frac{udu}{a^2+u^2} = \frac{1}{2}\ \ln\left|a^2+u^2\right| + c$
2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\ \arctan\left(\frac{u}{a}\right) + c$
3. $\displaystyle\int \frac{udu}{\sqrt{a^2+u^2}} = \sqrt{a^2+u^2} + c$
4. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left|u+\sqrt{a^2+u^2}\right| + c$
5. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{1}{a}\ \ln\left|\frac{\sqrt{a^2+u^2}}{u} - \frac{a}{u}\right| + c$
6. $\displaystyle\int \sqrt{a^2+u^2}\ du = \frac{1}{2}\ u\sqrt{a^2+u^2} + \frac{a^2}{2}\ \ln\left|u+\sqrt{a^2+u^2}\right| + c$

Theorem 6.5.3 *(u^2-a^2 Forms) Suppose that* $u = a\sec t,\ a > 0$. *Then*

$$du = a\sec t\tan t\ dt,\ u^2-a^2 = a^2\tan^2 t,\ \sqrt{u^2-a^2} = a\tan t, t = arcsec\left(\frac{u}{a}\right),$$

$$\sin t = \frac{\sqrt{u^2-a^2}}{u},\ \cos t = \frac{a}{u},\ \tan t = \frac{\sqrt{u^2-a^2}}{a},$$

$$\csc t = \frac{u}{\sqrt{u^2-a^2}},\ \sec t = \frac{u}{a},\ \cot t = \frac{a}{\sqrt{u^2-a^2}}.$$

graph

Proof. The proof of this theorem is left as an exercise.

The following integration formulas are valid:

1. $\displaystyle\int \frac{udu}{u^2 - a^2} = \frac{1}{2} \ln\left|u^2 - a^2\right| + c$

2. $\displaystyle\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln\left|\frac{u-a}{u+a}\right| + c$

3. $\displaystyle\int \frac{udu}{\sqrt{u^2 - a^2}} = \sqrt{u^2 - a^2} + c$

4. $\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + c$

5. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec}\left(\frac{u}{a}\right) + c$

6. $\displaystyle\int \sqrt{u^2 - a^2}\, du = \frac{1}{2}\, u\sqrt{u^2 - a^2} - \frac{a^2}{2} \ln\left|u + \sqrt{u^2 - a^2}\right| + c$

Exercises 6.5 Prove each of the following formulas:

1. $\displaystyle\int \frac{u\, du}{a^2 - u^2} = -\frac{1}{2} \ln|a^2 - u^2| + C$

2. $\displaystyle\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln\left|\frac{a-u}{a+u}\right| + C$

3. $\displaystyle\int \frac{u\, du}{\sqrt{a^2 - u^2}} = -\sqrt{a^2 - u^2} + C$

4. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C,\ a > 0$

5. $\displaystyle\int \frac{du}{u\sqrt{a^2 - u^2}} = \frac{1}{a} \ln\left|\frac{a}{u} - \frac{\sqrt{a^2 - u^2}}{u}\right| + C$

6. $\displaystyle\int \sqrt{a^2-u^2}\;du = \frac{a^2}{2}\ \arcsin\left(\frac{u}{a}\right) + \frac{1}{2}\ u\sqrt{a^2-u^2} + C,\ a>0$

7. $\displaystyle\int \frac{u\ du}{a^2+u^2} = \frac{1}{2}\ \ln\left|a^2+u^2\right| + C$

8. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\ \arctan\left(\frac{u}{a}\right) + C$

9. $\displaystyle\int \frac{u\ du}{\sqrt{a^2+u^2}} = \sqrt{a^2+u^2} + C$

10. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left|u+\sqrt{a^2+u^2}\right| + C$

11. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{1}{a}\ \ln\left|\frac{\sqrt{a^2+u^2}}{u} - \frac{a}{u}\right| + C$

12. $\displaystyle\int \sqrt{a^2+u^2}\;du = \frac{1}{2}\ u\sqrt{a^2+u^2} + \frac{a^2}{2}\ \ln\left|u+\sqrt{a^2+u^2}\right| + C$

13. $\displaystyle\int \frac{u\ du}{u^2-a^2} = \frac{1}{2}\ \ln\left|u^2-a^2\right| + C$

14. $\displaystyle\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ \ln\left|\frac{u-a}{u+a}\right| + C$

15. $\displaystyle\int \frac{u\ du}{\sqrt{u^2-a^2}} = \sqrt{u^2-a^2} + C$

16. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$

17. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\ \operatorname{arcsec}\left(\frac{u}{a}\right) + C$

18. $\displaystyle\int \sqrt{u^2-a^2}\;du = \frac{1}{2}\ u\sqrt{u^2-a^2} - \frac{a^2}{2}\ \ln\left|u+\sqrt{u^2-a^2}\right| + C$

Evaluate each of the following integrals:

19. $\int \frac{x\ dx}{\sqrt{4-x^2}}$

20. $\int \frac{dx}{\sqrt{4-x^2}}$

21. $\int \frac{x\ dx}{4-x^2}$

22. $\int \frac{dx}{4-x^2}$

23. $\int \frac{x\ dx}{9+x^2}$

24. $\int \frac{dx}{9+x^2}$

25. $\int \frac{x\ dx}{\sqrt{9+x^2}}$

26. $\int \frac{dx}{\sqrt{9+x^2}}$

27. $\int \frac{x\ dx}{x^2-16}$

28. $\int \frac{dx}{x^2-16}$

29. $\int \frac{x\ dx}{\sqrt{x^2-16}}$

30. $\int \frac{dx}{\sqrt{x^2-16}}$

31. $\int \frac{dx}{x\sqrt{x^2-4}}$

32. $\int \frac{dx}{x\sqrt{9-x^2}}$

33. $\int \frac{dx}{x\sqrt{x^2+16}}$

34. $\int \sqrt{9-x^2}\ dx$

35. $\int \sqrt{4-9x^2}$

36. $\int \frac{x^2}{\sqrt{1-x^2}}\ dx$

37. $\int \frac{x^2}{\sqrt{4+x^2}}\ dx$

38. $\int \frac{x^2}{\sqrt{x^2-16}}\ dx$

39. $\int \frac{dx}{(9+x^2)^2}$

40. $\int \frac{dx}{(9-x^2)^2}$

41. $\int \frac{dx}{(x^2-16)^2}$

42. $\int \frac{dx}{(4+x^2)^{3/2}}$

43. $\int \frac{\sqrt{4+x^2}}{x}$

44. $\int \frac{\sqrt{x^2-4}}{x}\ dx$

45. $\int \frac{dx}{x^2\sqrt{x^2+4}}$

46. $\int \frac{dx}{x^2\sqrt{4-x^2}}$

47. $\int \frac{dx}{x^2\sqrt{x^2-4}}$

48. $\int \frac{dx}{x^2-2x+5}$

49. $\int \frac{dx}{x^2-4x+12}$

50. $\int \frac{dx}{\sqrt{4x-x^2}}$

51. $\int \frac{dx}{\sqrt{x^2-4x+12}}$

52. $\int \frac{dx}{4x-x^2}$

53. $\int \frac{dx}{\sqrt{x^2-2x+5}}$

54. $\int \frac{x\ dx}{x^2-4x-12}$

55. $\int \frac{x\ dx}{\sqrt{x^2-2x+5}}$

56. $\int \frac{x}{x^2+4x+13}\ dx$

57. $\int (5-4x-x^2)^{1/2} dx$

58. $\displaystyle\int \frac{2x+7}{x^2+4+13}\,dx$ 59. $\displaystyle\int \frac{x+3}{\sqrt{x^2+2x+5}}\,dx$ 60. $\displaystyle\int \frac{dx}{\sqrt{4x^2-1}}$

61. $\displaystyle\int \frac{x+4}{\sqrt{9x^2+16}}\,dx$ 62. $\displaystyle\int \frac{x+2}{\sqrt{16-9x^2}}\,dx$ 63. $\displaystyle\int \frac{e^{2x}dx}{(5-e^{2x}+e^{4x})^{1/2}}$

64. $\displaystyle\int \frac{e^{3x}dx}{(e^{6x}+4e^{3x}+3)^{1/2}}$

6.6 Integration by Partial Fractions

A polynomial with real coefficients can be factored into a product of powers of linear and quadratic factors. This fact can be used to integrate rational functions of the form $P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials that have no factors in common. If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, then by long division we can express the rational function by

$$\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}$$

where $q(x)$ is the quotient and $r(x)$ is the remainder whose degree is less than the degree of $Q(x)$. Then $Q(x)$ is factored as a product of powers of linear and quadratic factors. Finally $r(x)/Q(x)$ is split into a sum of fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

and

$$\frac{B_1x+c_1}{ax^2+bx+c} + \frac{B_2x+c_2}{(ax^2+bxc)^2} + \cdots + \frac{B_mx+c_m}{(ax^2+bx+c)^m}.$$

Many calculators and computer algebra systems, such as Maple or Mathematica, are able to factor polynomials and split rational functions into partial fractions. Once the partial fraction split up is made, the problem of integrating a rational function is reduced to integration by substitution using linear or trigonometric substitutions. It is best to study some examples and do some simple problems by hand.

Exercises 6.6 Evaluate each of the following integrals:

1. $\displaystyle\int \frac{dx}{(x-1)(x-2)(x+4)}$

2. $\displaystyle\int \frac{dx}{(x-4)(10+x)}$

3. $\displaystyle\int \frac{dx}{(x-a)(x-b)}$

4. $\displaystyle\int \frac{dx}{(x-a)(b-x)}$

5. $\displaystyle\int \frac{dx}{(x^2+1)(x^2+4)}$

6. $\displaystyle\int \frac{dx}{(x-1)(x^2+1)}$

7. $\displaystyle\int \frac{2x\ dx}{x^2-5x+6}$

8. $\displaystyle\int \frac{x\ dx}{(x+3)(x+4)}$

9. $\displaystyle\int \frac{x+1}{(x+2)(x^2+4)}\ dx$

10. $\displaystyle\int \frac{(x+2)dx}{(x+3)(x^2+1)}$

11. $\displaystyle\int \frac{2\ dx}{(x^2+4)(x^2+9)}$

12. $\displaystyle\int \frac{dx}{(x^2-4)(x^2-9)}$

13. $\displaystyle\int \frac{x^2\ dx}{(x^2+4)(x^2+9)}$

14. $\displaystyle\int \frac{x\ dx}{(x^2-4)(x^2-9)}$

15. $\displaystyle\int \frac{dx}{x^4-16}$

16. $\displaystyle\int \frac{x\ dx}{x^4-81}$

6.7 Fractional Power Substitutions

If the integrand contains one or more fractional powers of the form $x^{s/r}$, then the substitution, $x = u^n$, where n is the least common multiple of the denominators of the fractional exponents, may be helpful in computing the integral. It is best to look at some examples and work some problems by hand.

Exercises 6.7 Evaluate each of the following integrals using the given substitution.

1. $\displaystyle\int \frac{4x^{3/2}}{1+x^{1/3}}\ dx;\ x = u^6$

2. $\displaystyle\int \frac{dx}{1+x^{1/3}};\ x = u^3$

3. $\displaystyle\int \frac{dx}{\sqrt{1+e^{2x}}}$; $u^2 = 1 + e^{2x}$

4. $\displaystyle\int \frac{dx}{x\sqrt{x^3-8}}$; $u^2 = x^3 - 8$

Evaluate each of the following by using an appropriate substitution:

5. $\displaystyle\int \frac{x\ dx}{\sqrt{x+2}}$

6. $\displaystyle\int \frac{x^2 dx}{\sqrt{x+4}}$

7. $\displaystyle\int \frac{1}{4+\sqrt{x}}\ dx$

8. $\displaystyle\int \frac{x\ dx}{1+\sqrt{x}}$

9. $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}$

10. $\displaystyle\int \frac{x^{2/3}}{8+x^{1/2}}$

11. $\displaystyle\int \frac{1}{x^{2/3}+1}\ dx$

12. $\displaystyle\int \frac{dx}{1+\sqrt{x}}$

13. $\displaystyle\int \frac{x\ dx}{1+x^{2/3}}$

14. $\displaystyle\int \frac{1+\sqrt{x}}{2+\sqrt{x}}\ dx$

15. $\displaystyle\int \frac{1-\sqrt{x}}{1+x^{3/2}}\ dx$

16. $\displaystyle\int \frac{1+\sqrt{x}}{1-x^{3/2}}\ dx$

6.8 Tangent $x/2$ Substitution

If the integrand contains an expression of the form $(a+b\sin x)$ or $(a+b\cos x)$, then the following theorem may be helpful in evaluating the integral.

Theorem 6.8.1 *Suppose that* $u = \tan(x/2)$. *Then*

$$\sin x = \frac{2u}{1+u^2},\ \cos x = \frac{1-u^2}{1+u^2}\ \text{ and }\ dx = \frac{2}{1+u^2}\ du.$$

Furthermore,

$$\int \frac{dx}{a+b\sin x} = \int \frac{(2/(1+u^2))du}{a+b\left(\frac{2u}{1+u^2}\right)} = \int \frac{2du}{a(1+u^2)+2bu}$$
$$\int \frac{dx}{a+b\cos x} = \int \frac{(2/(1+u^2))du}{a+b\left(\frac{1-u^2}{1+u^2}\right)} = \int \frac{2du}{a(1+u^2)+b(1-u^2)}.$$

Proof. The proof of this theorem is left as an exercise.

Exercises 6.8

1. Prove Theorem 6.8.1

Evaluate the following integrals:

2. $\displaystyle\int \frac{dx}{2+\sin x}$

3. $\displaystyle\int \frac{dx}{\sin x+\cos x}$

4. $\displaystyle\int \frac{dx}{\sin x-\cos x}$

5. $\displaystyle\int \frac{dx}{2\sin x+3\cos x}$

6. $\displaystyle\int \frac{dx}{2-\sin x}$

7. $\displaystyle\int \frac{dx}{3+\cos x}$

8. $\displaystyle\int \frac{dx}{3-\cos x}$

9. $\displaystyle\int \frac{\sin x\ dx}{\sin x+\cos x}$

10. $\displaystyle\int \frac{\cos x\ dx}{\sin x-\cos x}$

11. $\displaystyle\int \frac{(1+\sin x)dx}{(1-\sin x)}$

12. $\displaystyle\int \frac{1-\cos x}{1+\cos x}\ dx$

13. $\displaystyle\int \frac{2-\cos x}{2+\cos x}\ dx$

14. $\displaystyle\int \frac{2+\cos x}{2-\sin x}\ dx$

15. $\displaystyle\int \frac{2-\sin x}{3+\cos x}\ dx$

16. $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

6.9 Numerical Integration

Not all integrals can be computed in the closed form in terms of the elementary functions. It becomes necessary to use approximation methods. Some of the simplest numerical methods of integration are stated in the next few theorems.

Theorem 6.9.1 (Midpoint Rule) *If f, f' and f'' are continuous on $[a, b]$, then there exists some c such that $a < c < b$ and*

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{f''(c)}{24}\,(b-a)^3.$$

Proof. The proof of this theorem is omitted.

Theorem 6.9.2 (Trapezoidal Rule) *If f, f' and f'' are continuous on $[a, b]$, then there exists some c such that $a < c < b$ and*

$$\int_a^b f(x)dx = (b-a)\left[\frac{1}{2}\,(f(a)+f(b))\right] - \frac{f''(c)}{12}\,(b-a)^3.$$

Proof. The proof of this theorem is omitted.

Theorem 6.9.3 (Simpson's Rule) *If $f, f', f'', f^{(3)}$ and $f^{(4)}$ are continuous on $[a, b]$, then there exists some c such that $a < c < b$ and*

$$\int_a^b f(x)dx = \frac{b-a}{6}\,\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{f^{(4)}(c)}{2880}\,(b-a)^5.$$

These basic numerical formulas can be applied on each subinterval $[x_i, x_{i+1}]$ of a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of the interval $[a, b]$ to get composite numerical methods. We assume that $h = (b-a)/n,\ x_i = a + ih,\ i = 0, 1, 2, \cdots, n$.

Proof. The proof of this theorem is omitted.

Theorem 6.9.4 (Composite Trapezoidal Rule) *If f, f' and f'' are continuous on $[a, b]$, then there exists some c such that $a < c < b$ and*

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right] - \frac{b-a}{12}\,h^2 f''(c).$$

Proof. The proof of this theorem is omitted.

Theorem 6.9.5 (Composite Simpson's Rule) *If $f, f', f'', f^(3)$ and $f^{(4)}$ are continuous on $[a, b]$, then there exists some c such that $a < c < b$ and*

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 2\sum_{i=1}^{n/2-1} f(x_{2i}) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + f(b)\right] - \frac{b-a}{180}\,h^4 f^{(4)}(c).$$

where n is an even natural number.

Proof. The proof of this theorem is omitted.

Remark 22 In practice, the composite Trapezoidal and Simpson's rules can be applied when the value of the function is known at the subdivision points $x_i, i = 0, 1, 2, \cdots, n$.

Exercises 6.9 Approximate the value of each of the following integrals for a given value of n and using

(a) Left-hand end point approximation: $\sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$

(b) Right-hand end point approximation: $\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$

(c) Mid point approximation: $\sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)(x_i - x_{i-1})$

(d) Composite Trapezoidal Rule

(e) Composite Simpson's Rule

1. $\int_1^3 \frac{1}{x}\, dx,\ n = 10$

2. $\int_2^4 \frac{1}{\sqrt{x}}\, dx,\ n = 10$

3. $\int_0^1 \frac{1}{1+\sqrt{x}}\, dx,\ n = 10$

4. $\int_1^2 \frac{1}{1+x^2}\, dx,\ n = 10$

5. $\int_0^1 \frac{1+\sqrt{x}}{1+x}\, dx,\ n = 10$

6. $\int_0^2 x^3\, dx,\ n = 10$

7. $\int_0^2 (x^2 - 2x)\, dx,\ n = 10$

8. $\int_0^1 (1+x^2)^{1/2} dx,\ n = 10$

9. $\int_0^1 (1+x^3)^{1/2} dx,\ n = 10$

10. $\int_0^1 (1+x^4)^{1/2} dx,\ n = 10$